MATH 2040031-Graph Theory @Lanzhou University

General Information

Instructors: Shou-Jun XU et al.

Class period: total 36 Sections, 2 sections per week.

Textbook: Introduction to Graph theory, 5th Edition, by Robin, J. Wilson, Pearson, 2014.

Reference books:

1. Introduction to Graph Theory, 2nd Edition, by Douglas B. West, Prentice Hall, 2001.
2. Introduction to graph and hypergraph theory, by Vitaly I. Voloshin, Nova, 2009.
3. A textbook of graph theory, 2nd Edition, by R. Balakrishnan, K. Ranganathan, Springer, 2012.

Course Syllabus

Week 1: Introduction

Definitions: *vertex, edge, graph, degree of a vertex, multiple edge, loop, simple graph, directed graph (digraph), arc, walk, path, cycle, Eulerian graph, Hamiltonian graph, connected, disconnected, tree, planar graph.*

Theorems, Lemmas and Corollaries: four-colour theorem; marriage problem.

Week 2: Definitions and examples

Definitions: *simple graph, vertex (node or point), edge (line), vertex-set, edge-set, join, multiple edge, loop, general graph, graph, isomorphic, union, connected, disconnected, component, adjacent, incident, degree, isolated vertex, end-vertex, degree sequence, subgraph, complement, adjacent matrix, incident matrix, null graph, complete graph, cycle graph, path, wheel, regular graph, regular of degree, r-regular, cubic graph, pe-*

tersen graph, platonic graph, bipartite graph, complete bipartite graph, k-cube.

Theorems, Lemmas and Corollaries: handshaking lemma; no odd number of odd degrees.

Week 3: Definitions and examples

Definitions: *directed graph, vertex, arc, vertex-set, edge-family, underlying graph, simple digraph, isomorphic, connected, disconnected, join, adjacent, incident, out-degree, in-degree, adjacent matrix, incident matrix, tournament.*

Theorems, Lemmas and Corollaries: handshaking dilemma.

Week 4: Infinite graphs and Three puzzles

Definitions: *infinite graph, countable graph, degree of an infinite graph, locally finite, locally countable, self-complementary, vertex space of graph G, line graph, automorphism, automorphism graph, three puzzles: the eight-circles problem, six people at a party, the four-cubes problems.*

Theorems, Lemmas and Corollaries: every connected locally countable infinite graph is a countable graph; every connected locally finite graph is a countable graph.

Week 5: Paths and cycles

Definitions: *walk, initial vertex, final vertex, length, trail, path, closed, cycle, triangle, disconnected set, cut set, bridge, disconnected set, edge-connectivity, k-edge-connected, separating, cut-vertex, connectivity, k-connected.*

Theorems, Lemmas and Corollaries: a graph is bipartite if and only if every cycle of G has even length; any simple graph with n vertices and more than $\frac{1}{2}(n\text{-}1)(n\text{-}2)$ edges is connected; a graph G is k-edge-connected if and only if any two distinct vertices of G are joined by at least k paths, no two of which have any edges in common; menger's theorem; a connected graph G is orientable if and only if each edge of G lies in at least one cycle; Konig's lemma.

Week 6: Paths and cycles

Definitions: *strong connected orientable, orientation, Eulerian, Eulerian trial, semi-Eulerian, Eulerian digraph, infinite Eulerian graph, Hamiltonian cycle, Hamiltonian graph, semi-Hamiltonian, Hamiltonian digraph, weighted graph, weight.*

Theorems, Lemmas and Corollaries: a graph whose minimum degree at least 2 contains a cycle; Euler's theorem; a connected graph is Eulerian if and only if its set of edges can be split up into edge-disjoint cycles; a strongly connected digraph is Eulerian if and only if for each vertex V of D, outdeg(v) =indeg(v); Dirac's theorem; Ore's theorem; every tournament is "nearly Hamiltonian".

Week 7: Paths, cycles, applications

Definitions and Algorithms: *Efficient algorithms; applications: the shortest path problem, the critical path problem; the Chinese postman problem; the travelling salesman proble; Fleury's algorithm.*

Week 8: Trees

Definitions: *tree, spanning tree, cycle rank, cutset rank, the complete of a spanning tree, fundamental set of cycles associated with a spanning tree, fundamental set of cycles of a graph, fundamental set of cutset associated with a spanning tree.*

Theorems, Lemmas and Corollaries: characterizations of a tree (equivalent definitions); each cutset (cycle) of a graph has an edge in common with a spanning tree of the graph (the complement of a spanning tree of the graph); Cayley's theorem; the number of spanning trees of K_n is n^{n-2}; matrix-tree theorem.

Week 9: Trees

Definitions: *the minimum connector problem, searching tree, root, depth-first search, breadth-first search.*

Theorems, Lemmas and Corollaries: a solution of the minimum connector problem; a braced rectangular framework is rigid if and only if the corresponding bipartite graph is connected.

Algorithms: greedy algorithm, depth-first search algorithm, breadth-first search algorithm, Kirchhoff's laws.

Week 10: Planarity

Definitions: *planar graph, plane drawing, plane graph, crossing number, homeomorphic, contractible, outerplanar, face, infinite face, polyhedral graph, thickness.*

Theorems, Lemmas and Corollaries: $K_{3,3}$ and K_5 are non-planar; Kuratowski's theorem; a graph is planar if and only if it contains no subgraph contractible to $K_{3,3}$ or K_5; if G is a countable graph, every finite subgraph of which is planar, then G is planar; Euler's formula; for every polyhedral graph, we have $n - m + f = 2$; for every simple connected planar graph with n (≥ 3) vertices and m edges, we have $m \leq 3n - 6$;

every simple planar graph contains a vertex of degree at most 5.

Week 11: Planarity

Definitions: *(geometric) dual, abstract dual, genus, toroidal graph, face.*

Theorems, Lemmas and Corollaries: the relations between the numbers of vertices, edges and faces of connected planar graph and its geometric dual; planar graph is an abstract dual of its abstract dual graph; a graph is planar if and only if it has an abstract dual; the genus of a graph does not exceed the crossing number; let G be a connected graph of genus g with n vertices, m edges and f faces, then $n - m + f = 2 - 2g$; Ringel and Youngs' theorem.

Week 12: Colouring graph

Definitions: *k-colourable, k-chromatic, chromatic number, chromatic function, map, k-colourable-f, k-colourable-v.*

Theorems, Lemmas and Corollaries: the simple graph with largest vertex-degree Δ is $(\Delta + 1)$-colourable; Brooks's theorem; every simple planar graph is 6-colourable; five-colour theorem; every simple planar graph is 4-colourable; the chromatic function of a simple graph is a polynomial; a map is 2-colourable-f if and only if it is an Eulerian graph.

Week 13: Colouring graph

Definitions: *unavoidable set of configuration, reducible configuration, method of Kempe chains, reducible, Birkhoff diamond, chromatic index.*

Theorems, Lemmas and Corollaries: the Birkhoff diamond is reducible; Vizing's theorem; the four-colour theorem is equivalent to the statement that $\chi^{'}(G) = 3$ for each cubic map G; $\chi^{'}(K_{r,s}) = \max(r, s)$.

Week 14: Matching, marriage and Menger's theorem

Definitions: *marriage problem, complete matching, marriage condition, transversal, partial transversal, edge-disjoint path, vertex-disjoint path, vw-disconnected set, vw-separating set.*

Theorems, Lemmas and Corollaries: Hall's theorem; let E be a non-empty finite set, and let $\mathcal{F} = (S_1, S_2, ..., S_m)$ be a family of non-empty subsets of E, then $\mathcal{F}$ has a transversal if and only if the union of any k of the subsets S_i contains at least k elements, for $1 \leq k \leq m$; Menger's theorem; a graph G with at least $k+1$ vertices is k-connected if and only if any two distinct vertices of G are connected by at least k vertex-disjoint path; Menger's theorem implies Hall's theorem.

Week 15: Matching, marriage and Menger's theorem

Definitions: *capacity, source, sink, flow, zero flow, non-zero flow, saturated, unsaturated, value of the flow, maximum flow, capacity of a cut, maximum cut, flow-augmenting path.*

Theorems, Lemmas and Corollaries: max-flow min-cut theorem.

Week 16: Matroids

Definitions: *matroid, bases, rank, cycle matroid, vector matroid, independent, parallel element, isomorphic, trivial matroid, discrete matroid, k-uniform matroid, cycle matroid, graphic matroid, cutset matroid, co-graphic, planar matroid, bipartite matroid, Eulerian matroid, representable over a field, representable matroid, regular matroid, binary matroid, transversal matroid, Fano matroid, restriction, contraction, minor.*

Theorems, Lemmas and Corollaries: a matroid consists of a non-empty finite set E and an integer-valued function r defined on the set of subsets of E, satisfying: r(i) $0 \leq r(A) \leq |A|$, for each subsets A of E; r(ii) if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$; r(iii) for any A, B $\subseteq$ E, $r(A \bigcup B) + r(A \bigcap B) \leq r(A) + r(B)$.

Week 17: Matroids

Definitions: *dual matroid.*

Theorems, Lemmas and Corollaries: dual matroid is a matroid on E; if G is a connected graph, then $M^*(G) = (M(G))^*$; every cocycle of a matroid intersects every base; every cycle of a matroid intersects every cobase; if G^* is an abstract dual of a graph G, then $M(G^*)$ is isomorphic to $(M(G))^*$; Tutte's theorem; a matroid is planar if and only if it is regular and contains no minor isomorphic to $M(K_5)$, $M(K_{3,3})$ or their dual.

Week 18: Review